June 18-19, 2009, Redondo Beach, CA

Nonlinear Theory of Ion-Cyclotron Instability

Peter H. Yoon and Ta-Ming Fang

Massachusetts Technological Laboratory

Research supported by AFOSR contract FA9550-07-0053

- Linear Theory of Ion-Cyclotron Instability

$$D^{\pm}(k,\omega) = 0$$

$$= 1 - \frac{c^2k^2}{\omega^2} - \sum_j \frac{\omega_{pj}^2}{\omega^2} \int d\mathbf{v} \, \frac{(\omega - kv_z)F_j - (kv_\perp^2/2)(\partial F_j/\partial v_z)}{\omega - \omega_{cj} - kv_z}$$

- Bi-Maxwellian Distribution

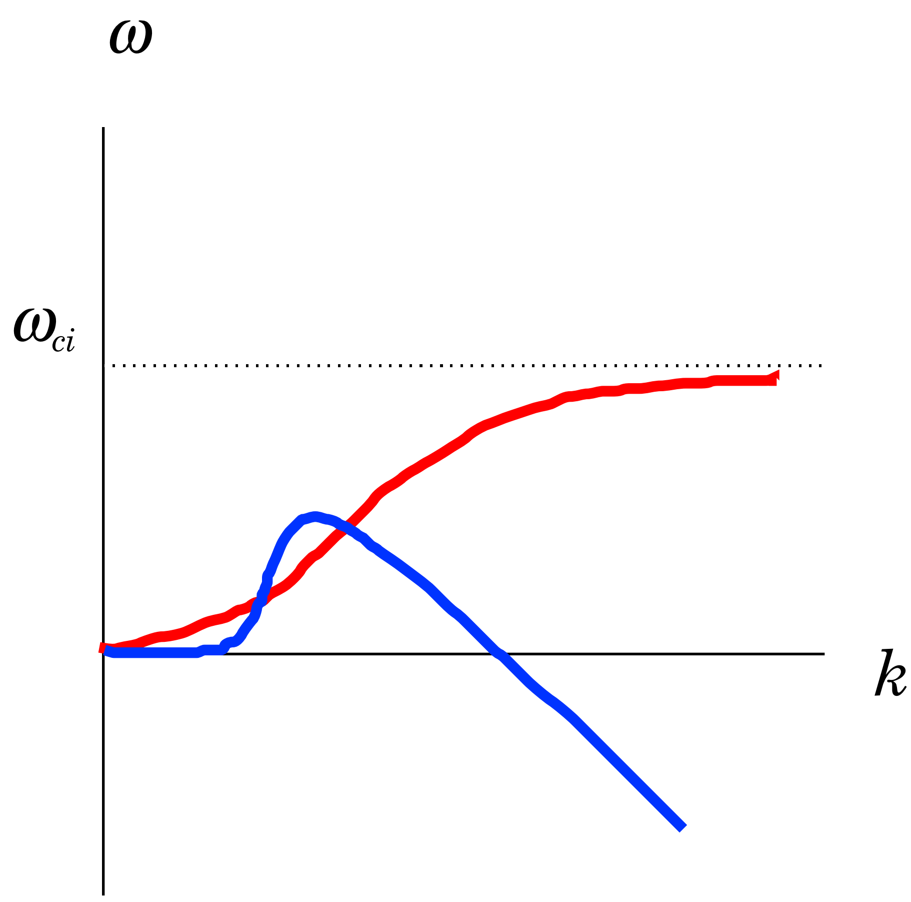
$$D^{\pm}(k,\omega) = 0$$

$$= 1 - \frac{c^2k^2}{\omega^2} + \sum_j \frac{\omega_{pj}^2}{\omega^2} \left[\frac{\omega}{kv_{Tj}} Z(\xi_j^{\pm}) - \left(1 - \frac{T_{j\perp}}{T_{j\parallel}}\right) [1 + \xi_j^{\pm} Z(\xi_j^{\pm})] \right]$$

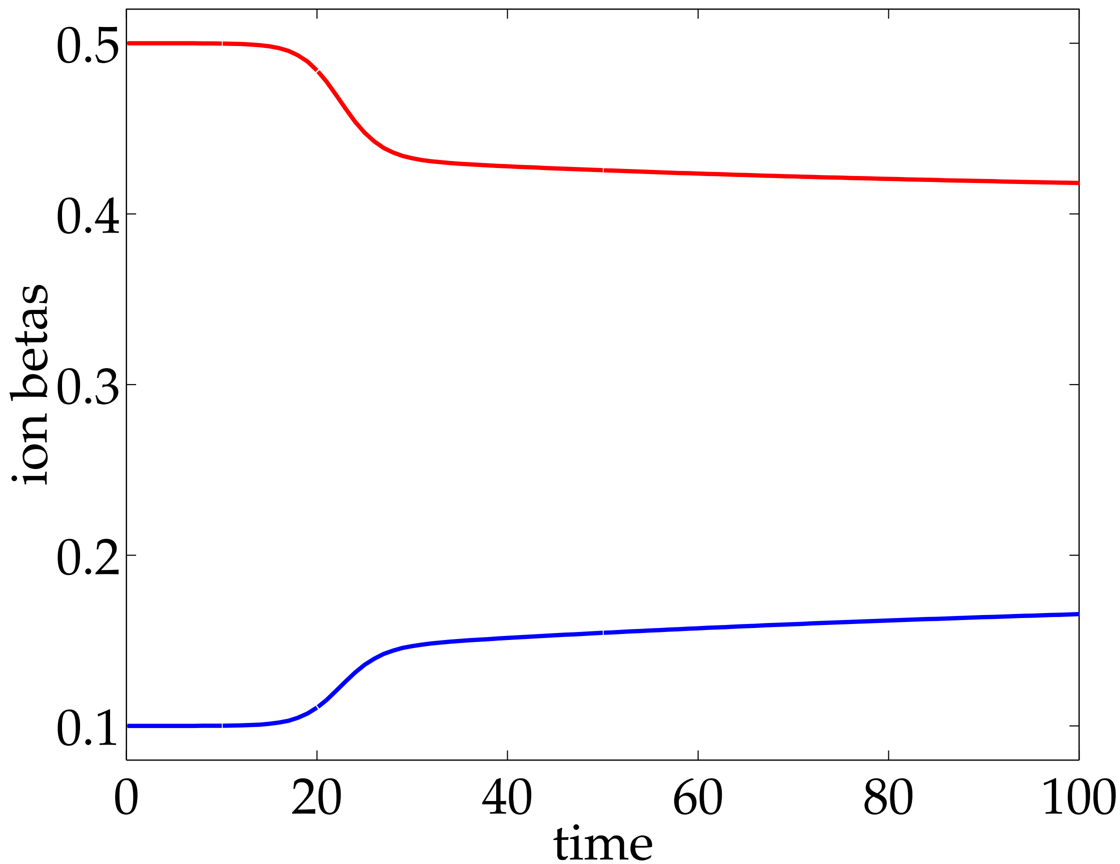
$$v_{Tj} = \left(\frac{2T_{j\parallel}}{m_j}\right)^{1/2}, \quad \xi_j^{\pm} = \frac{\omega \pm \omega_{cj}}{kv_{Tj}}$$

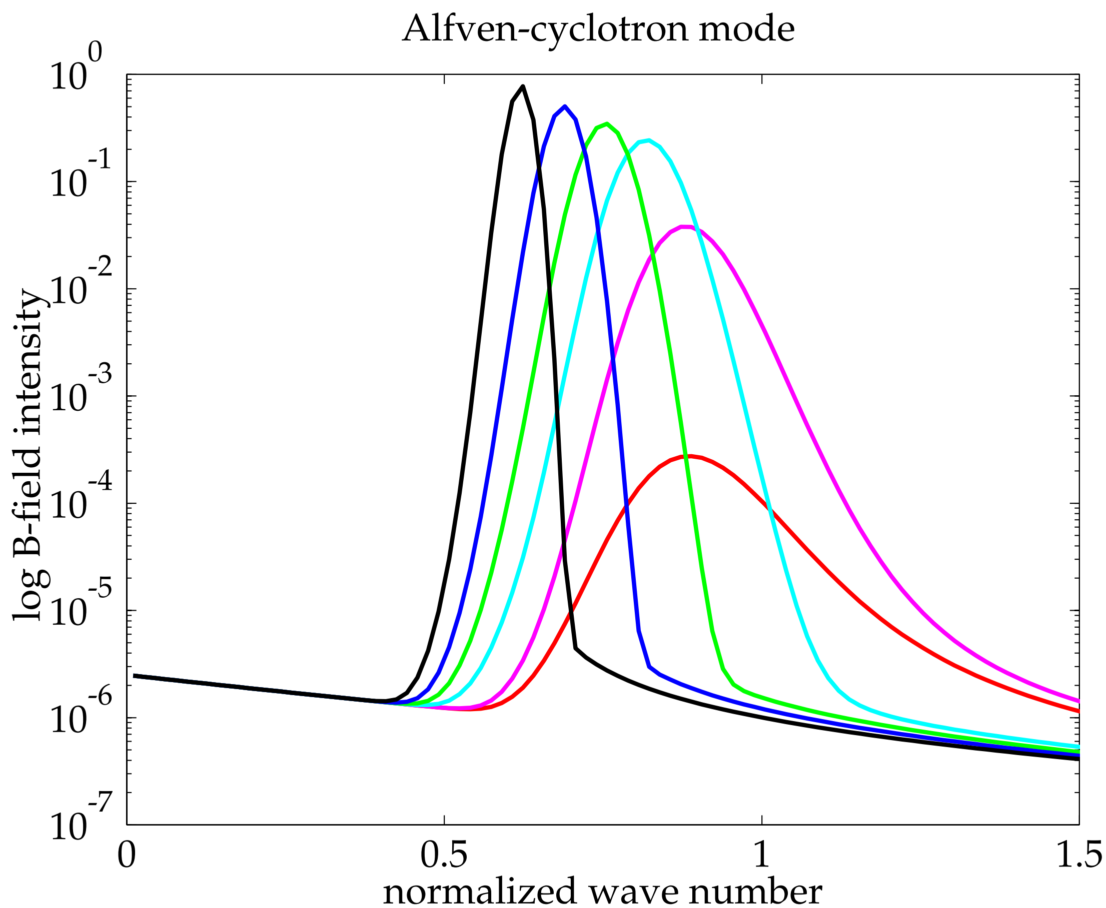
Ion-Cyclotron Instability

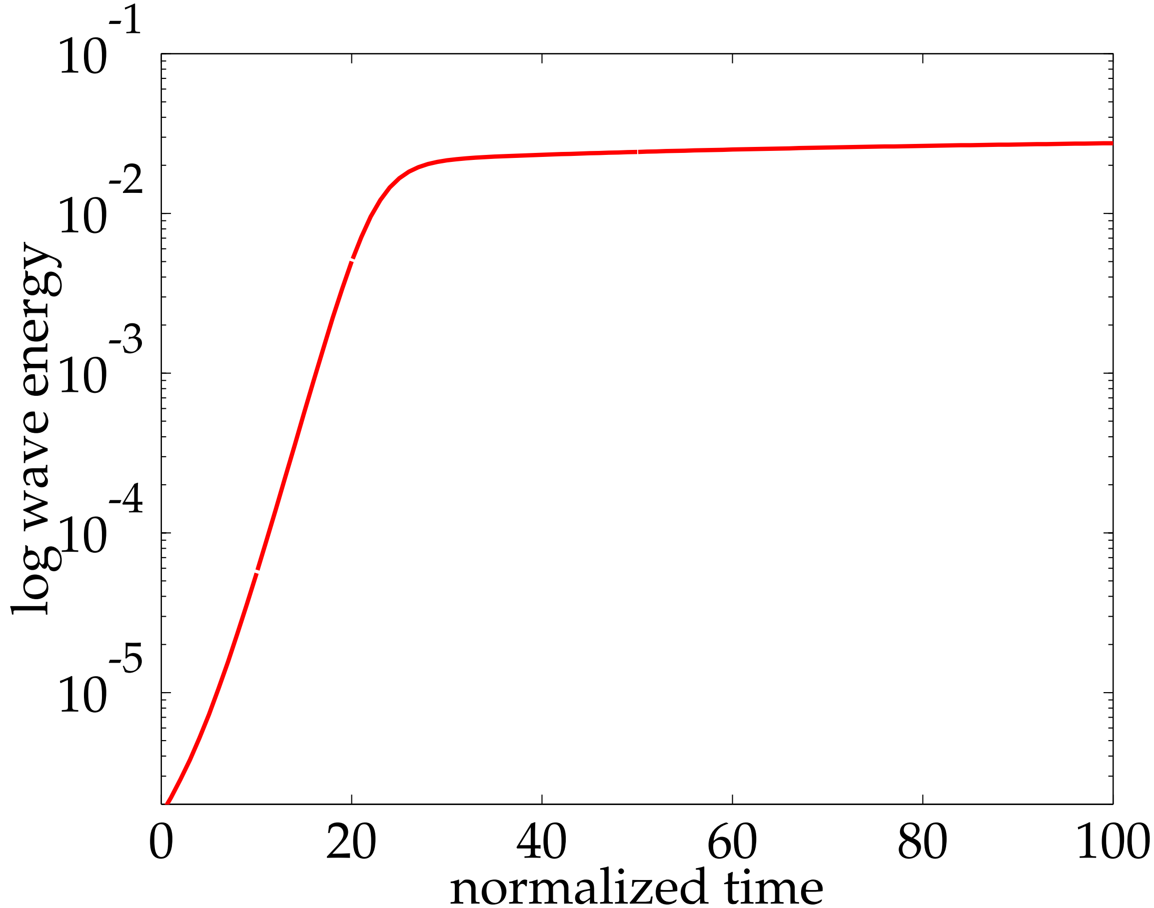
$$T_{i\perp}/T_{i\parallel} > 1$$



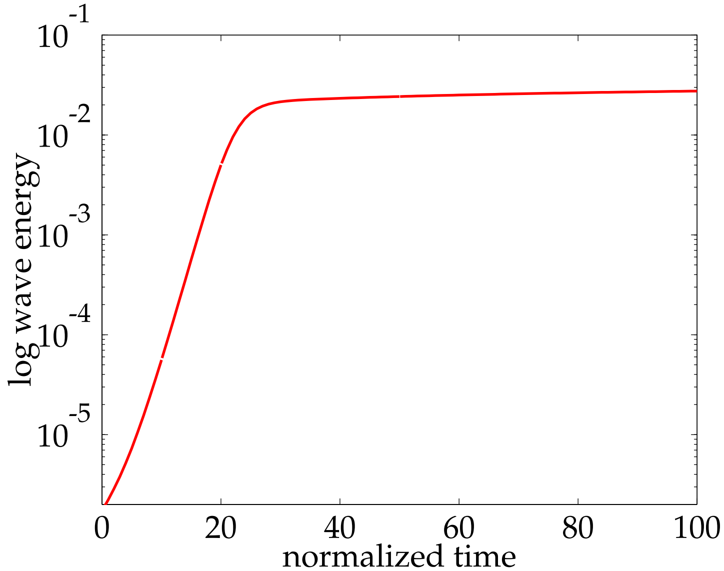
Quasilinear Theory of Ion-Cyclotron Instability

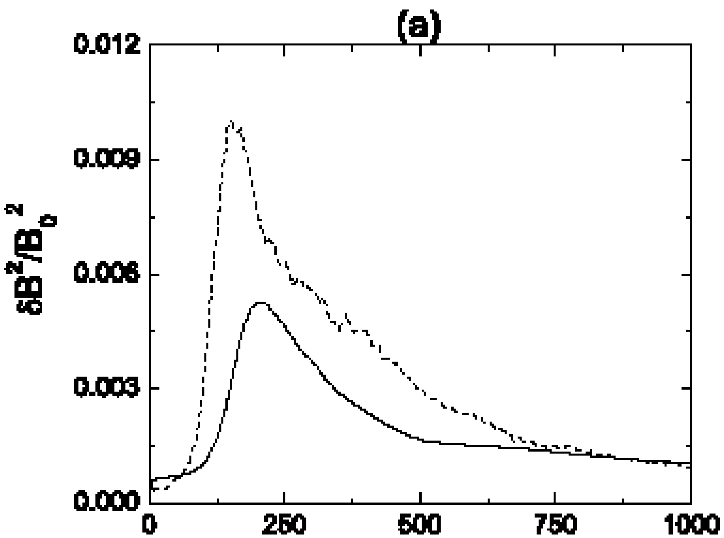



Alfven-cyclotron mode
log B-field intensity
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
0.5
1
1.5
normalized wave number



log wave energy
normalized time
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
20
40
60
80
100





Lu et al., Phys. Plasmas (2009)

Nonlinear Theory of Ion-Cyclotron Instability

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_a = 0,$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0,$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\sum_a e_a n_a \int d\mathbf{v}\,\mathbf{v}\, f_a.$$

Weak Turbulence Theory for Magnetized Plasmas

$$f_a = F_a + \delta f_a,$$

$$\mathbf{E} = \delta\mathbf{E},$$

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}.$$

Weak turbulence theory available in the literature was formulated largely by the Russians (Kadomtsev, Tsytovich, etc.), but they are applicable only for unmagnetized plasmas.

$$\frac{\partial F_a}{\partial t} = \frac{e_a}{m_a}\int d\mathbf{k}d\omega \left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\delta_{ij} + \frac{v_j k_i}{\omega}\right] < \delta E^j_{-\mathbf{k},-\omega}\delta f^a_{\mathbf{k},\omega} >,$$

$$\left[\delta_{ij} - \frac{c^2k^2}{\omega^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\right]\delta E^j_{\mathbf{k},\omega} = -\frac{4\pi i}{\omega}\sum_a e_a n_a \int d\mathbf{v}\; v_i \delta f^a_{\mathbf{k},\omega},$$

$$\omega_{ca}\frac{\partial \delta f^a_{\mathbf{k},\omega}}{\partial\varphi} + i(\omega - \mathbf{k}\cdot\mathbf{v})\delta f^a_{\mathbf{k},\omega} = \frac{e_a}{m_a}\delta E^i_{\mathbf{k},\omega}\left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\delta_{ij} + \frac{v_i k_j}{\omega}\right]\frac{\partial F_a}{\partial v_j}$$

$$+\frac{e_a}{m_a}\frac{\partial}{\partial v_i}\int d\mathbf{k}'d\omega'\left[\left(1-\frac{\mathbf{k}'\cdot\mathbf{v}}{\omega'}\right)\delta_{ij} + \frac{v_i k'_j}{\omega'}\right]$$

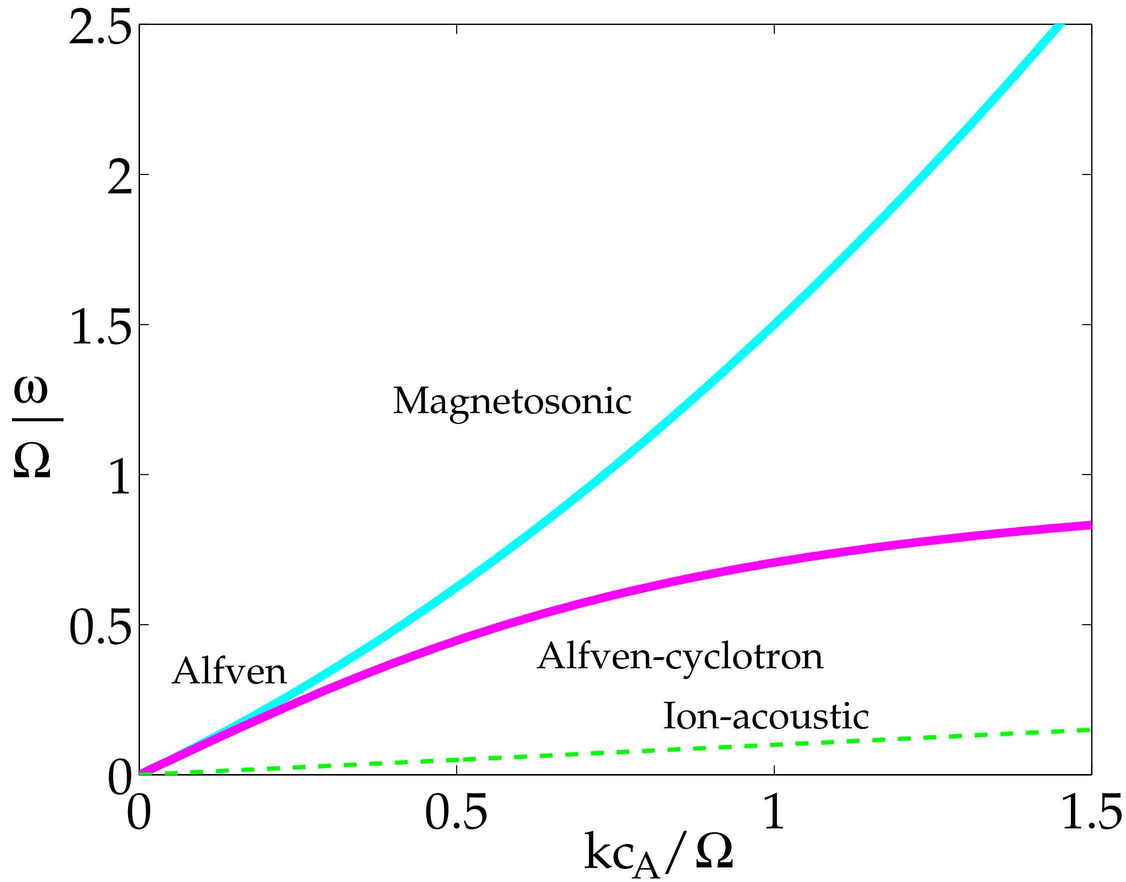
$$\times\left[\delta E^i_{\mathbf{k}',\omega'}\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - < \delta E^i_{\mathbf{k}',\omega'}\delta f^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'} >\right].$$

$$\frac{dT_\perp}{dt} = \sum_\pm \int dk M_k \left(1 - \frac{T_\perp}{T_\parallel} + \frac{\omega_k^\pm}{\omega_c}\frac{T_\perp}{T_\parallel} \right) e^{-(\omega_k^\pm - \omega_c)^2/k^2 v_\parallel^2} I_\pm(k),$$

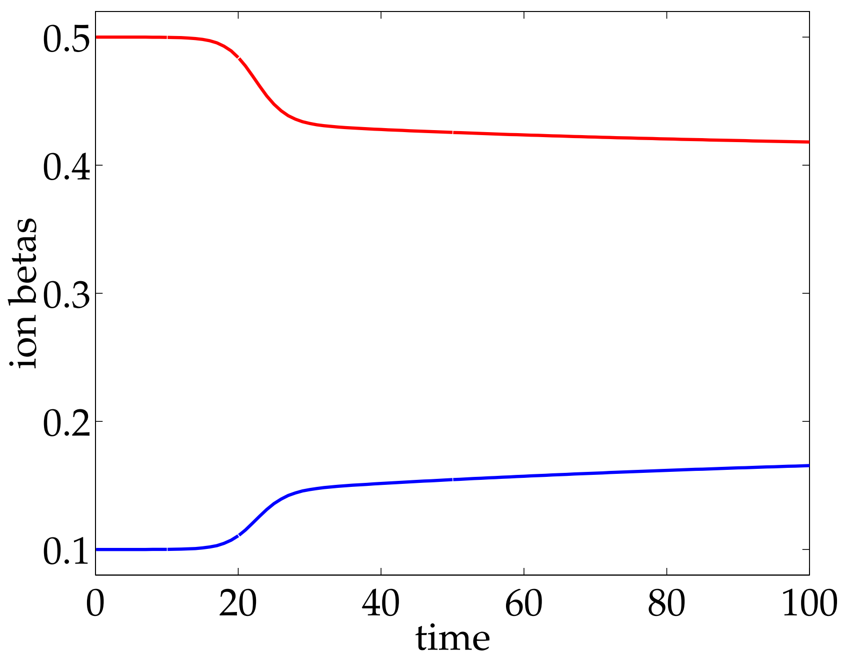
$$\frac{dT_\parallel}{dt} = -2\frac{dT_\perp}{dt} + \sum_\pm \int dk N_k \left(1 - \frac{T_\perp}{T_\parallel} + \frac{\omega_k^\pm}{\omega_c}\frac{T_\perp}{T_\parallel} \right) e^{-(\omega_k^\pm - \omega_c)^2/k^2 v_\parallel^2} I_\pm(k),$$

$$\frac{\partial I_\pm(k)}{\partial t} = 2\gamma_k^\pm I_\pm(k) + \int dk' V_{k,k'} \left[I_\pm(k') I_S(k-k') + \cdots \right],$$

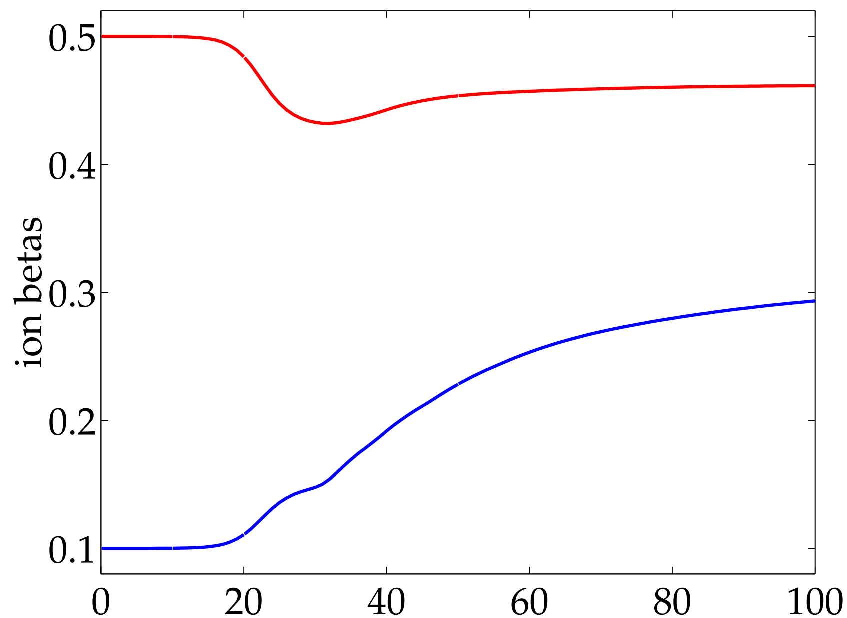
$$\frac{\partial I_S(k)}{\partial t} = 2\gamma_k^S I_S(k) + \int dk' U_{k,k'} \left[I_\pm(k') I_\pm(k-k') + \cdots \right].$$

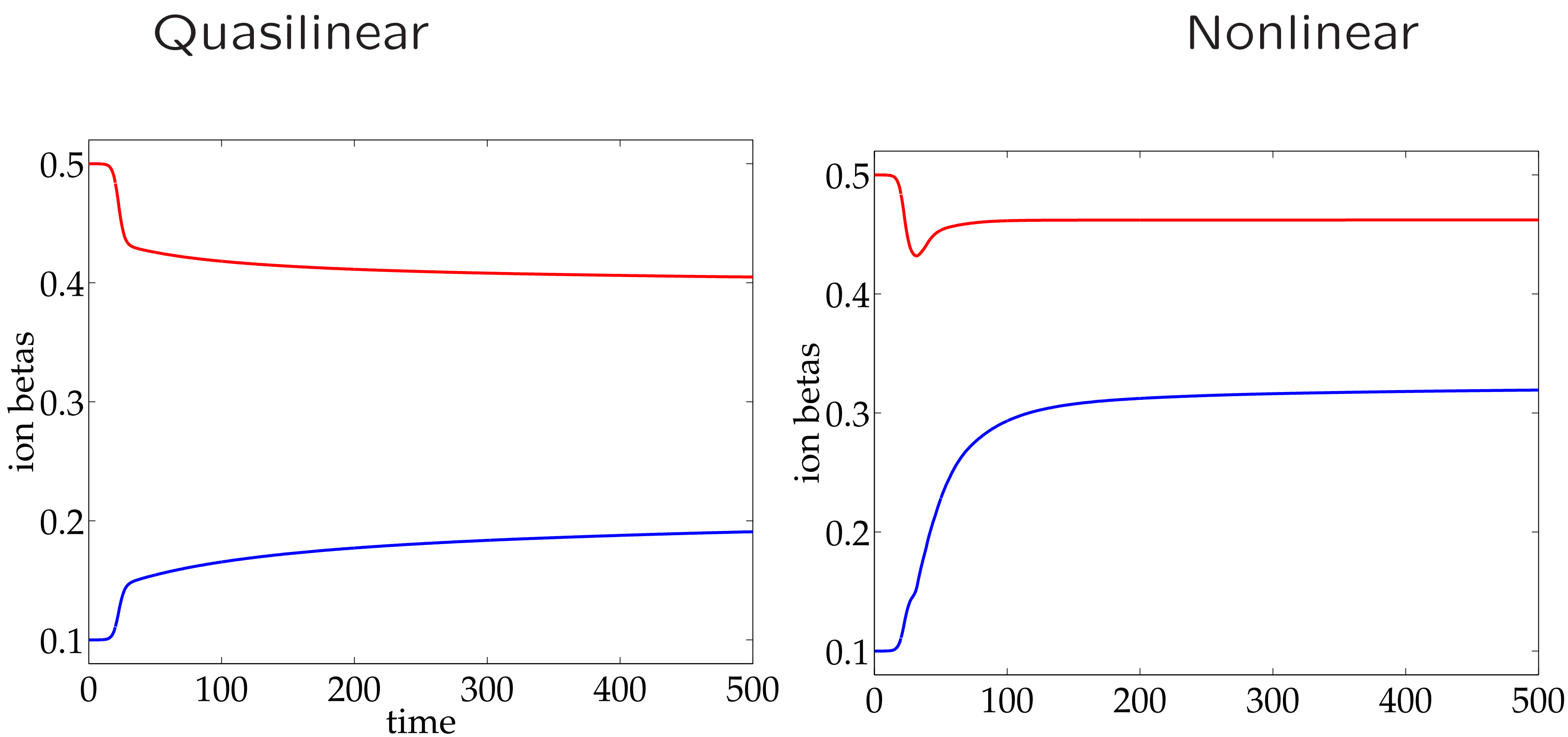

2.5
2
1.5
1
0.5
0
$\frac{\omega}{\Omega}$
Magnetosonic
Alfven
Alfven-cyclotron
Ion-acoustic
0
0.5
1
1.5
kc_A/Ω

Quasilinear

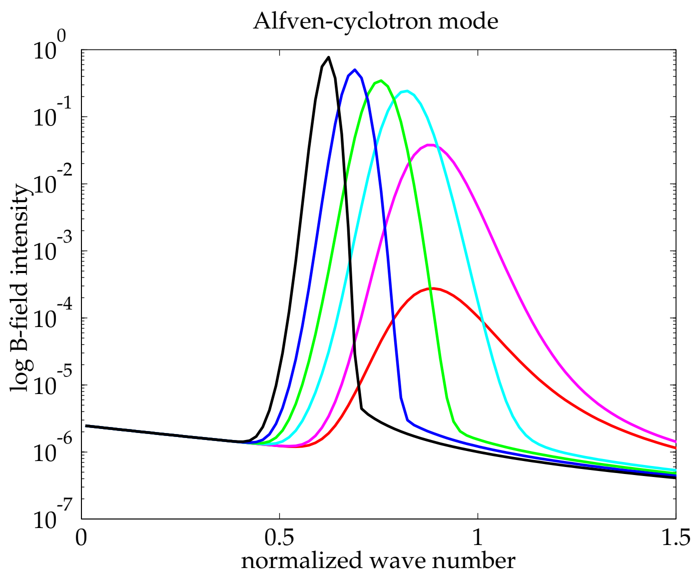


Nonlinear

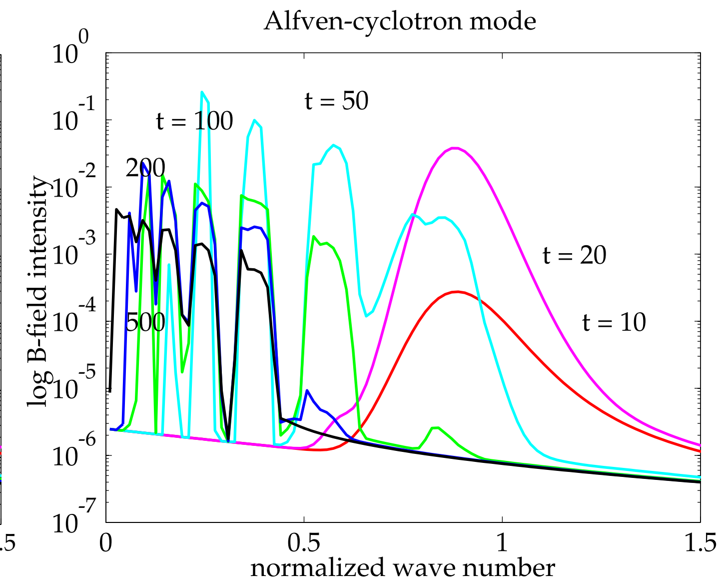




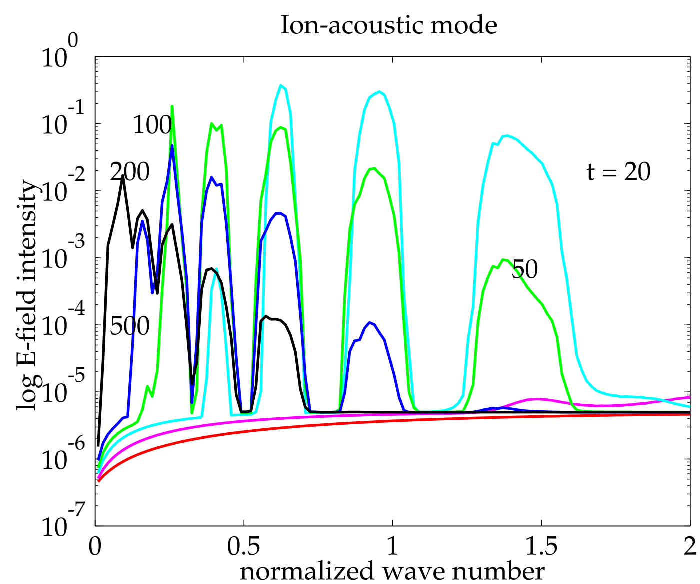
Quasilinear
Nonlinear
ion betas
time
0.5
0.4
0.3
0.2
0.1
0
100
200
300
400
500



Alfven-cyclotron mode
log B-field intensity
normalized wave number

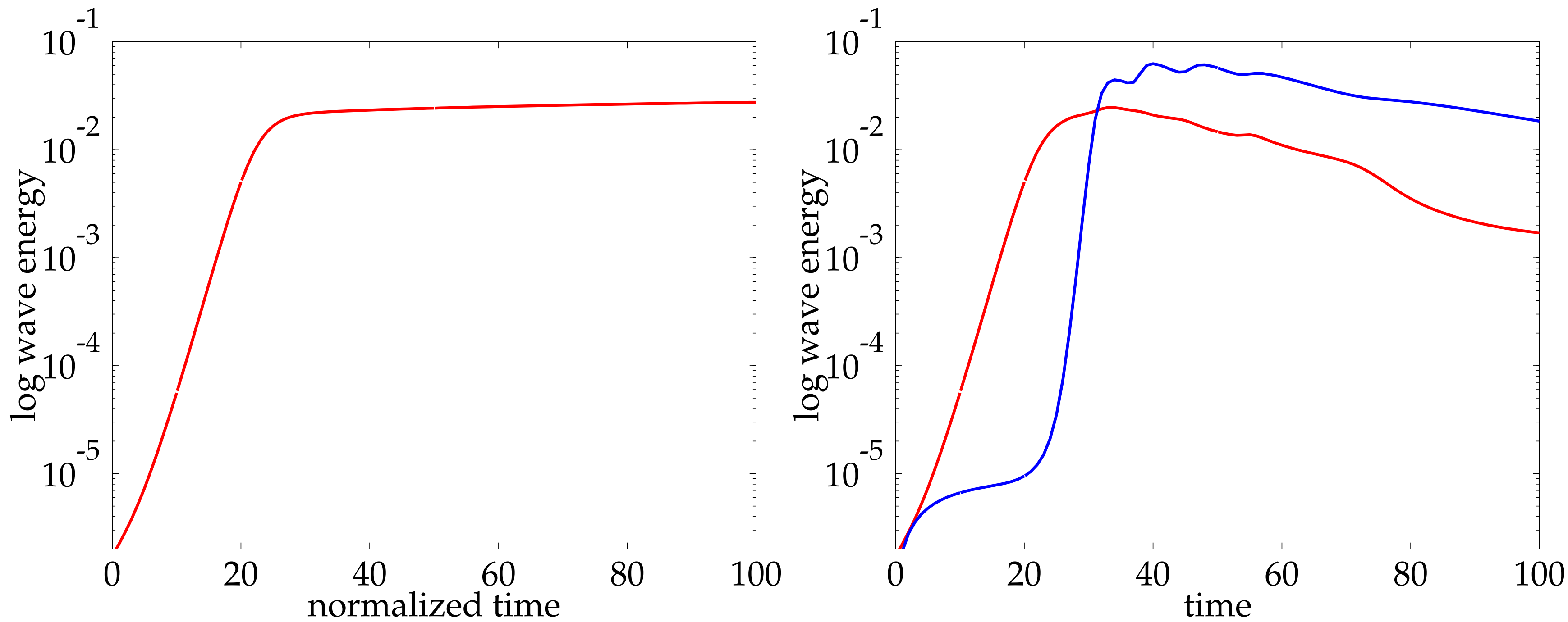


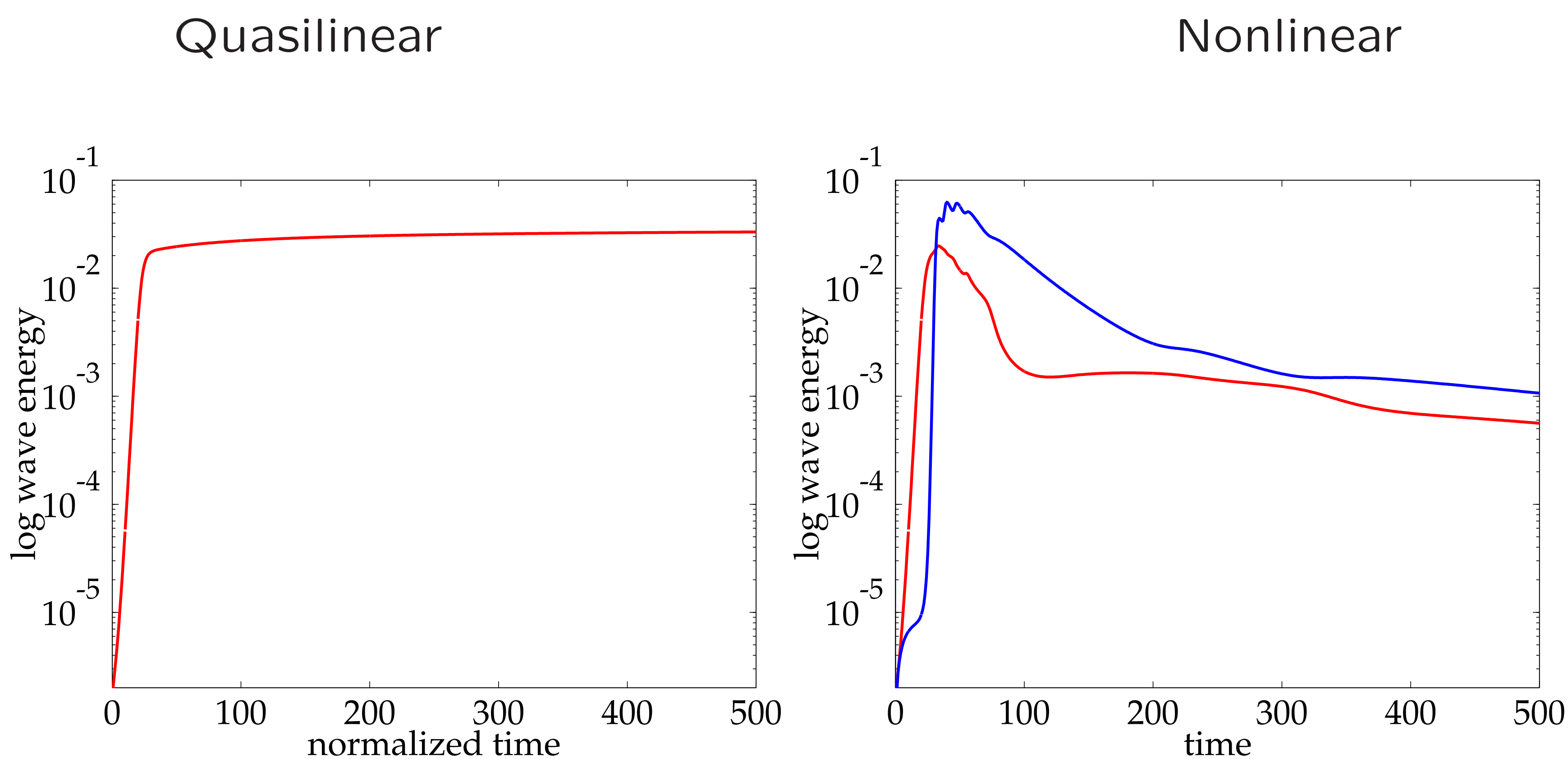
Alfven-cyclotron mode
t = 50
t = 100
200
500
t = 20
t = 10
log B-field intensity
normalized wave number



Ion-acoustic mode
100
200
500
50
t = 20
log E-field intensity
normalized wave number

Quasilinear Nonlinear





Quasilinear
Nonlinear
log wave energy
10^-1
10^-2
10^-3
10^-4
10^-5
0
100
200
300
400
500
normalized time
time

Conclusions and Discussion

- Nonlinear theory of ion-cyclotron instability based upon weak turbulence theory for magnetized plasmas leads to correct understanding of nonlinear physics.

- Need to benchmark against simulations.

- 2D effects need to be included.

- The basic formalism can be applied to other plasma instabilities operative in magnetized plasmas.

Appendix 1: Formal Kinetic Equations

Particle Kinetic Equation

$$\frac{\partial F_a}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{A}\, F_a) + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\vec{\vec{D}} \cdot \frac{\partial F_a}{\partial \mathbf{v}} \right),$$

$$A_\perp = \frac{e_a^2}{4\pi\, m_a} \frac{\Omega_i^2}{\omega_{pi}^2} \sum_\sigma \sum_{+,-} \int dk\, \frac{\mp \Omega_a}{\sigma \omega_k^\pm}\, g_\pm^\sigma(k)\, \delta(\sigma \omega_k^\pm - k v_\parallel \pm \Omega_a)\, v_\perp,$$

$$A_\parallel = \frac{e_a^2}{4\pi\, m_a} \sum_\sigma \int dk\, \left(\frac{\Omega_i^2}{\omega_{pi}^2} \sum_{+,-} \frac{k v_\perp}{\sigma \omega_k^\pm}\, g_\pm^\sigma(k)\, \delta(\sigma \omega_k^\pm - k v_\parallel \pm \Omega_a)\, v_\perp \right.$$

$$\left. + \,\delta(\sigma \omega_k^\parallel - k v_\parallel)\, h(k)\, v_\parallel \right),$$

$$D_{\perp\perp} = \frac{\pi\, e_a^2}{m_a^2} \sum_\sigma \int dk \ \sum_{+,-} \frac{\Omega_a^2}{(\omega_k^\pm)^2}\, I_\pm^\sigma(k)\ \delta(\sigma\omega_k^\pm - kv_\parallel \pm \Omega_a),$$

$$D_{\perp\parallel} = \frac{\pi\, e_a^2}{m_a^2} \sum_\sigma \int dk \ \sum_{+,-} \frac{\mp\Omega_a\, kv_\perp}{(\omega_k^\pm)^2}\, I_\pm^\sigma(k)\ \delta(\sigma\omega_k^\pm - kv_\parallel \pm \Omega_a),$$

$$D_{\parallel\parallel} = \frac{\pi\, e_a^2}{m_a^2} \sum_\sigma \int dk \ \left(\sum_{+,-} \frac{(kv_\perp)^2}{(\omega_k^\pm)^2}\, I_\pm^\sigma(k)\ \delta(\sigma\omega_k^\pm - kv_\parallel \pm \Omega_a) \right.$$

$$\left. + \, \delta(\sigma\omega_k^\parallel - kv_\parallel)\, I_\parallel^\sigma(k) \right).$$

Ion-Sound Wave Kinetic Equation

$$\frac{\partial I_{\|}^{\sigma}(k)}{\partial t} = \pi\,\sigma c_S\,h(k)\sum_a \omega_{pa}^2 \int d\mathbf{v}\;\delta(\sigma\omega_k^{\|} - k v_{\|})$$

$$\times\left(\frac{m_a}{4\pi^2}\,\sigma c_S\,h(k)\,F_a + \frac{\partial F_a}{\partial v_{\|}}\,I_{\|}^{\sigma}(k)\right)$$

$$+\,\frac{\pi\,e^2}{m_i^2}\,\sigma k c_S\,h(k)\sum_{+,-}\sum_{\sigma',\sigma''}\int dk'\;\delta(\sigma\omega_k^{\|} - \sigma'\omega_{k'}^{\mp} - \sigma''\omega_{k-k'}^{\pm})$$

$$\times\,\Big[v_{\pm}(k,k')\,\sigma k c_S\,h(k)\,I_{\mp}^{\sigma'}(k')I_{\pm}^{\sigma''}(k-k')$$

$$+\,u_{\pm}(k,k')\,\sigma'\omega_{k'}^{\mp}\,g_{\mp}^{\sigma'}(k')\,I_{\pm}^{\sigma''}(k-k')I_{\|}^{\sigma}(k)$$

$$-\,w_{\pm}(k,k')\,\sigma''\omega_{k-k'}^{\pm}\,g_{\pm}^{\sigma''}(k-k')\,I_{\mp}^{\sigma'}(k')I_{\|}^{\sigma}(k)\Big]\,,$$

$$v_{\pm}(k,k') = \frac{\omega_{pi}^4}{(\omega_{k'}^{\mp})^2\,(\omega_{k-k'}^{\pm})^2\,k^4 c_S^4}\left(\frac{\sigma'' k'\,\omega_{k-k'}^{\pm}}{\Omega_i \pm \sigma''\omega_{k-k'}^{\pm}} - \frac{\sigma'(k-k')\,\omega_{k'}^{\mp}}{\Omega_i \mp \sigma'\omega_{k'}^{\mp}}\right)^2,$$

$$u_{\pm}(k,k') = \frac{\omega_{pi}^2\,\Omega_i^2}{(\omega_{k'}^{\mp})^2\,(\omega_{k-k'}^{\pm})^2\,k^3 c_S^4}\left(\frac{\sigma'' k'\,\omega_{k-k'}^{\pm}}{\Omega_i \pm \sigma''\omega_{k-k'}^{\pm}} - \frac{\sigma'(k-k')\,\omega_{k'}^{\mp}}{\Omega_i \mp \sigma'\omega_{k'}^{\mp}}\right)$$

$$\times\ \left(\frac{\sigma\,(k-k')\,c_S}{\Omega_i} - \frac{\sigma''\omega_{k-k'}^{\pm}}{\Omega_i \pm \sigma''\omega_{k-k'}^{\pm}} + \frac{\sigma\,(k-k')\,c_S\,\Omega_i}{(\Omega_i \mp \sigma'\omega_{k'}^{\mp})\,(\Omega_i \pm \sigma''\omega_{k-k'}^{\pm})}\right),$$

$$w_{\pm}(k,k') = \frac{\omega_{pi}^2\,\Omega_i^2}{(\omega_{k'}^{\mp})^2\,(\omega_{k-k'}^{\pm})^2\,k^3 c_S^4}\left(\frac{\sigma'' k'\,\omega_{k-k'}^{\pm}}{\Omega_i \pm \sigma''\omega_{k-k'}^{\pm}} - \frac{\sigma'(k-k')\,\omega_{k'}^{\mp}}{\Omega_i \mp \sigma'\omega_{k'}^{\mp}}\right)$$

$$\times\ \left(\frac{\sigma k' c_S}{\Omega_i} - \frac{\sigma'\omega_{k'}^{\mp}}{\Omega_i \mp \sigma'\omega_{k'}^{\mp}} + \frac{\sigma k' c_S\,\Omega_i}{(\Omega_i \mp \sigma'\omega_{k'}^{\mp})\,(\Omega_i \pm \sigma''\omega_{k-k'}^{\pm})}\right),$$

Right/Left-Circularly Polarized EM Wave Kinetic Equation

$$\frac{\partial I_\pm^\sigma(k)}{\partial t} = \pi \frac{\Omega_i^2}{\omega_{pi}^2} g_\pm^\sigma(k) \sum_a \frac{\omega_{pa}^2}{\sigma\omega_k^\pm} \int d\mathbf{v}\; v_\perp^2\, \delta(\sigma\omega_k^\pm - kv_\parallel \pm \Omega_a)$$

$$\times \left[\frac{m_a}{4\pi^2} \frac{\Omega_i^2}{\omega_{pi}^2} \sigma\omega_k^\pm\, g_\pm^\sigma(k)\, F_a + \left(\mp\Omega_a \frac{\partial F_a}{v_\perp \partial v_\perp} + k\frac{\partial F_a}{\partial v_\parallel}\right) I_\pm^\sigma(k)\right]$$

$$+ \frac{2\pi\, e^2}{m_i^2} \frac{\Omega_i^2}{(\omega_k^\pm)^2} \sigma\omega_k^\pm\, g_\pm^\sigma(k) \sum_{\sigma',\sigma''} \int dk'\; \delta(\sigma\omega_k^\pm - \sigma'\omega_{k'}^\pm - \sigma''\omega_{k-k'}^\parallel)$$

$$\times \Big[V_\pm(k,k')\, \sigma\omega_k^\pm\, g_\pm^\sigma(k)\, I_\pm^{\sigma'}(k')\, I_\parallel^{\sigma''}(k-k')$$

$$- U_\pm(k,k')\, \sigma'\omega_{k'}^\pm\, g_\pm^{\sigma'}(k')\, I_\parallel^{\sigma''}(k-k')\, I_\pm^\sigma(k)$$

$$+ W_\pm(k,k')\, \sigma''\omega_{k-k'}^\parallel\, h(k-k')\, I_\pm^{\sigma'}(k')\, I_\pm^\sigma(k)\Big],$$

$$V_\pm(k,k') = \frac{\Omega_i^2}{(\omega_{k'}^\pm)^2\,(k-k')^2 c_S^4}\left(\frac{\sigma'' k' c_S}{\Omega_i} - \frac{\sigma' \omega_{k'}^\pm}{\Omega_i \pm \sigma' \omega_{k'}^\pm}\right.$$

$$\left. + \frac{\sigma'' k' c_S\, \Omega_i}{(\Omega_i \pm \sigma \omega_k^\pm)(\Omega_i \pm \sigma' \omega_{k'}^\pm)}\right)^2,$$

$$U_\pm(k,k') = \frac{\Omega_i^2}{(\omega_{k'}^\pm)^2\,(k-k')^2\, c_S^4}\left(\frac{\sigma'' k' c_S}{\Omega_i} - \frac{\sigma' \omega_{k'}^\pm}{\Omega_i \pm \sigma' \omega_{k'}^\pm}\right.$$

$$\left. + \frac{\sigma'' k' c_S\, \Omega_i}{(\Omega_i \pm \sigma \omega_k^\pm)\,(\Omega_i \pm \sigma' \omega_{k'}^\pm)}\right)$$

$$\times\ \left(\frac{\sigma'' k c_S}{\Omega_i} - \frac{\sigma \omega_k^\pm}{\Omega_i \pm \sigma \omega_k^\pm} + \frac{\sigma'' k c_S\, \Omega_i}{(\Omega_i \pm \sigma \omega_k^\pm)\,(\Omega_i \pm \sigma' \omega_{k'}^\pm)}\right),$$

$$W_{\pm}(k,k') = \frac{\omega_{pi}^2}{(\omega_{k'}^{\pm})^2\,(k-k')^3\,c_S^4}\left(\frac{\sigma' k\,\omega_{k'}^{\pm}}{\Omega_i \pm \sigma'\omega_{k'}^{\pm}} - \frac{\sigma k'\,\omega_{k}^{\pm}}{\Omega_i \pm \sigma\omega_{k}^{\pm}}\right)$$

$$\times\,\left(\frac{\sigma'' k' c_S}{\Omega_i} - \frac{\sigma'\omega_{k'}^{\pm}}{\Omega_i \pm \sigma'\omega_{k'}^{\pm}} + \frac{\sigma'' k' c_S\,\Omega_i}{(\Omega_i \pm \sigma\omega_{k}^{\pm})\,(\Omega_i \pm \sigma'\omega_{k'}^{\pm})}\right).$$