We Can Determine Radiation Belt Time Evolution Operators from Multipoint Correlations

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Overview

- Radiation belt time evolution is linear in the phase space density (PSD) (e.g., diffusion is a linear operator)
- The time-forward operator relates the spatial correlation function to the spatiotemporal correlation function of the system
- PSD measurements from two or more spacecraft can give us the spatial and spatiotemporal correlation functions we need to fully specify the time-forward operator (even if it's not diffusion) under specified geomagnetic conditions θ (e.g., Kp)
- Subsequent manipulations can extract the diffusion coefficients from the linear operator

$$\dot{f} = \hat{D}_t f + S_t$$

$$\vec{f}_{t+\Delta t} \approx \left(\underline{\underline{I}} + \Delta t \underline{\underline{\hat{D}}}_t\right)\vec{f}_t + \Delta t \vec{S}_t$$

$$\vec{g}_t \approx \vec{f}_t + \left\langle \vec{f}_t \middle| \theta \right\rangle \qquad \left\langle \vec{g}_t \middle| \theta \right\rangle = 0$$

$$\vec{g}_{t+\Delta t} \approx \left(\underline{\underline{I}} + \Delta t \underline{\underline{\hat{D}}}_\theta\right)\vec{g}_t - \Delta t \underline{\underline{\hat{D}}}_\theta \bar{f}_\theta + \Delta t \vec{S}_\theta$$

$$\left\langle \vec{g}_{t+\Delta t} \vec{g}_t^T \middle| \theta \right\rangle \approx \left(\underline{\underline{I}} + \Delta t \underline{\underline{\hat{D}}}_\theta\right)\left\langle \vec{g}_t \vec{g}_t^T \middle| \theta \right\rangle$$

$$\underline{\underline{R}}_\theta^T \approx \left(\underline{\underline{I}} + \Delta t \underline{\underline{\hat{D}}}_\theta\right)\underline{\underline{\Sigma}}_\theta$$

$$\underline{\underline{\hat{D}}}_\theta \approx \left(\underline{\underline{R}}_\theta^T \underline{\underline{\Sigma}}_\theta^{-1} - \underline{\underline{I}}\right)\frac{1}{\Delta t}$$

1-D is Simple

- In the 1-D case, we can directly invert the diffusion operator to obtain the diffusion coefficient from any one of the eigenvectors of D_θ to within a constant offset.
- Each eigenvector gives a distinct estimate of the diffusion coefficient, which should help estimate the error.
- See V.3 “Quadrature (Spatial)” in Schulz and Lanzerotti
- *Yes, Virginia, everything you ever wanted to know about the radiation belts really is in S&L*

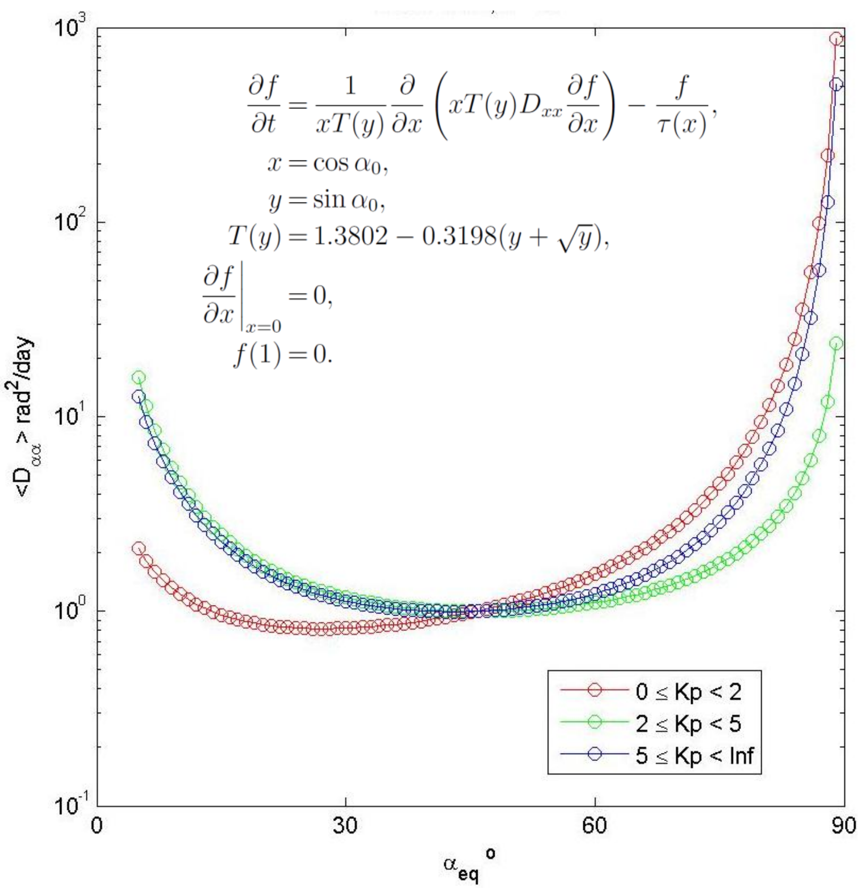
$$\underline{\underline{\hat{D}}}_\theta \vec{v}_k = \lambda_k \vec{v}_k$$

$$\frac{\partial f}{\partial t} = \frac{1}{G}\frac{\partial}{\partial Q}\left(GD\frac{\partial f}{\partial Q}\right) + S$$

$$\frac{\partial v_k}{\partial t} = \frac{1}{G}\frac{\partial}{\partial Q}\left(GD\frac{\partial v_k}{\partial Q}\right) = \lambda_k v_k$$

$$D = D_{Q=0} + \frac{\int_0^Q G\lambda_k v_k dQ'}{G\frac{\partial v_k}{\partial Q}}$$

Example in 1-D



- Using the covariance and lag covariance of electron pitch angle distributions at L=4.5, 800 keV, we can infer the pitch-angle diffusion coefficient
- This analysis uses the 2nd eigenmode of the inferred diffusion operator to compute a diffusion coefficient
- The solution is normalized to 1/day at 40 degrees (a necessity of the eigenmode approach)
- At "active" times (Kp > 5), pitch angle scattering is relatively stronger at low pitch angles, and relatively weaker at higher pitch angles compared to quiet times
- Electromagnetic ion cyclotron (EMIC) waves, which are expected only at active times, might explain this difference
- We can do this in 1-D (α) or 2-D (E,α) with a single spacecraft (like CRRES or a single RBSP) that can measure the full Energy-Angle distribution
- To do it in 3-D (E,K,L*), we would need at least 2 spacecraft (RBSP!) – but we could capture the all-important radial transport dynamics

3-D is harder

- The diffusion operator is the sum of 6 operators (3 “diagonal” diffusion terms and 3 “off-diagonal” diffusion terms)
- One way to tackle the problem is to set it up as an inversion on the first M>=6 eigenvectors
- This is a “big” numerical problem. If there are N grid points, then we have to invert a 6Nx6N matrix
- Another solution would be to set up a parameterized diffusion tensor, and then determine the (small number of) free parameters by a fitting procedure

$$\frac{\partial f}{\partial t} = \frac{1}{G}\sum_i \frac{\partial}{\partial Q_i}\left(G D_{ij} \sum_j \frac{\partial f}{\partial Q_j} \right) + S$$

$$\frac{\partial v_k}{\partial t} = \frac{1}{G}\sum_i \frac{\partial}{\partial Q_i}\left(G D_{ij} \sum_j \frac{\partial v_k}{\partial Q_j} \right) = \lambda_k v_k$$

$$\sum_{ij}\left[\frac{1}{G}\frac{\partial}{\partial Q_i}\left(G\frac{\partial v_k}{\partial Q_j} \right) + \frac{\partial v_k}{\partial Q_j}\frac{\partial}{\partial Q_i} \right] D_{ij} = \lambda_k v_k$$

$$\hat{B}_{ijk} = \left[\frac{1}{G}\frac{\partial}{\partial Q_i}\left(G\frac{\partial v_k}{\partial Q_j} \right) + \frac{\partial v_k}{\partial Q_j}\frac{\partial}{\partial Q_i} \right]$$

$$\begin{bmatrix} \underline{\underline{\hat{B}}}_{111} & 2\underline{\underline{\hat{B}}}_{121} & 2\underline{\underline{\hat{B}}}_{131} & \underline{\underline{\hat{B}}}_{221} & 2\underline{\underline{\hat{B}}}_{231} & \underline{\underline{\hat{B}}}_{331} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{\underline{\hat{B}}}_{11M} & 2\underline{\underline{\hat{B}}}_{12M} & 2\underline{\underline{\hat{B}}}_{13M} & \underline{\underline{\hat{B}}}_{22M} & 2\underline{\underline{\hat{B}}}_{23M} & \underline{\underline{\hat{B}}}_{33M} \end{bmatrix} \begin{bmatrix} \vec{D}_{11} \\ \vec{D}_{12} \\ \vec{D}_{13} \\ \vec{D}_{22} \\ \vec{D}_{23} \\ \vec{D}_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{v}_1 \\ \vdots \\ \lambda_M \vec{v}_M \end{bmatrix}$$

Summary

- Spatial covariance and spatiotemporal covariance of global PSD distributions can be used to infer the linear time evolution operator of the radiation belts
- Measuring those covariances requires at least 2 spacecraft capable of measuring PSD (or unidirectional differential flux)
- RBSP can do it